SYMMETRIES OF JULIA SETS OF POLYNOMIAL SKEW PRODUCTS ON $\mathbb{C}^2$

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Abstract. We consider the symmetries of Julia sets of polynomial skew products on $\mathbb{C}^2$, which are birationally conjugate to rotational products. Our main results give the classification of the polynomial skew products whose Julia sets have infinitely many symmetries.

1. Introduction

Any kind of Julia sets of a polynomial map can have symmetries. We say that a Julia set has symmetries if some nonelementary transformations preserve it. Beardon [1] investigated the symmetries of the Julia sets of polynomials on $\mathbb{C}$. He considered conformal functions as symmetries. To generalize the results in one-dimension to those in higher dimensions, we [3] previously investigated the symmetries of the Julia sets of nondegenerate polynomial skew products on $\mathbb{C}^2$. We defined the Julia sets as the supports of the Green measures, which are compact, and considered suitable polynomial automorphisms as symmetries. In this paper, we investigate the symmetries of Julia sets of polynomial skew products on $\mathbb{C}^2$, which generalize some of these previous results in [1] and [3]. We define the Julia sets by the fiberwise Green functions, which are close to the supports of the Green measures. However, the Julia sets may no longer be compact.

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$. More precisely, let $p(z) = a_\delta z^\delta + O(z^{\delta-1})$ and $q(z,w) = q_z(w) = b_d(z)w^d + O_z(w^{d-1})$. We assume that $\delta \geq 2$ and $d \geq 2$. Our results are as follows. First, we define the centroids of f as defined in [1], and show that the symmetries of the Julia set of f are birationally conjugate to rotational products. The tools of the proof are the fiberwise Green and Böttcher functions of f, which also relate to the centroids of f. Next, under some assumptions, we characterize the group of symmetries by functional equations including the iterates of f.

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The assumptions are, for example, the normality of f and the special form of the polynomial b_d. The normality of f, assuming f is in normal form, means that the centroids are at the origin. Finally, we classify the polynomial skew products whose Julia sets have infinitely many symmetries. Our main results claim that these maps are classified into four types.

This paper is organized into five sections, including this one. In Section 2, we briefly recall the dynamics of polynomials and the relevant results on the symmetries of the Julia sets of polynomials. In Section 3, we recall the dynamics of polynomial skew products. In particular, we review the existence of the fiberwise Green and Böttcher functions, and give the definition of Julia sets. The study of the symmetries of Julia sets begins in Section 4. We show that the symmetries are birationally conjugate to rotational products, and characterize the group of symmetries by functional equations. This section concludes with several examples, which include polynomial skew products that are semiconjugate to polynomial products whose Julia sets have infinitely many symmetries. We classify the polynomial skew products whose Julia sets have infinitely many symmetries in Section 5. We have two main theorems for the classification: the case when the map is in normal form and the case when it is not in normal form.

2. Symmetries of Julia sets of polynomials

In this section, we recall the dynamics of polynomials on $\mathbb{C}$ and the relevant results on the symmetries of the Julia sets of polynomials.

Let $p(z) = a_\delta z^\delta + a_{\delta-1} z^{\delta-1} + \cdots + a_0$ be a polynomial of degree $\delta \geq 2$. We denote by $p_2 p_1$ the composition of polynomials p_1 and p_2: $p_2 p_1(z) = p_2(p_1(z))$. Let p^n be the n-th iterate of p. A useful tool for the study of the dynamics of p is the Green function of p,

$$G_p(z) = \lim_{n \to \infty} \delta^{-n} \log^+ |p^n(z)|.$$

It is well known that the limit G_p is a nonnegative, continuous and subharmonic function on $\mathbb{C}$. By definition, $G_p(p(z)) = \delta G_p(z)$. Moreover, G_p is harmonic on $\mathbb{C} \setminus K_p$ and zero on K_p, where $K_p = \{z : \{p^n(z)\}_{n \geq 1} \text{ bounded}\}$, and $G_p(z) = \log|z| + \frac{1}{\delta-1} \log|a_\delta| + o(1)$ as $z \to \infty$. This is the Green function for K_p with a pole at infinity, determined only by the the compact set K_p. This function induces the Böttcher function φ_p defined near infinity such that $\varphi_p(z) = z + O(1)$ as $z \to \infty$, $\log|c\varphi_p(z)| = G_p(z)$, where $c = \sqrt[\delta-1]{a_\delta}$, and $\varphi_p(p(z)) = a_\delta(\varphi_p(z))^\delta$.

Let us recall some objects and results of the symmetries of the Julia sets of polynomials on $\mathbb{C}$. For further details, see [1]. We define the

Julia set J_p of p as the boundary ∂K_p, and consider conformal functions as the symmetries of J_p. Since J_p is compact, such functions are conformal Euclidean isometries. Hence the group of the symmetries of J_p is defined by

$$\Sigma_p = \{\sigma \in E : \sigma(J_p) = J_p\},$$

where $E = \{\sigma(z) = c_1 z + c_2 : |c_1| = 1, c_1, c_2 \in \mathbb{C}\}$. The centroid of p is defined by

$$\zeta = \frac{-a_{\delta-1}}{\delta a_\delta}.$$

If the solutions of $p(z) = Z$ are $z_1, z_2, \cdots, z_\delta$, then $p(z) = a_\delta(z - z_1)(z - z_2)\cdots(z - z_\delta) + Z$ and so the center of gravity of the points z_j coincides with ζ. It is known that each symmetry σ is a rotation about the centroid of p.

Proposition 2.1 ([1, Theorem 5]). *For any symmetry σ in Σ_p, there exists μ in the unit circle S^1 such that $\sigma(z) = \mu(z - \zeta) + \zeta$.*

We can characterize Σ_p by the unique equation.

Proposition 2.2 ([1, Lemma 7]). *It follows that $\Sigma_p = \{\sigma \in E : p\sigma = \sigma^\delta p\}$.*

By Proposition 2.1, the group Σ_p is identified with a subgroup of the unit circle S^1. This group is trivial, finite cyclic or infinite. We have a sufficient and necessary condition for Σ_p to be infinite.

Proposition 2.3 ([1, Lemma 4]). *The group Σ_p is infinite if and only if p is affinely conjugate to z^δ, or equivalently, if J_p is a circle. In this case, Σ_p consists of all rotations about ζ.*

We say that p is in normal form if $a_\delta = 1$ and $a_{\delta-1} = 0$, so that the centroid is at the origin. We may assume that p is in normal form without loss of generality because p is conjugate to a polynomial in normal form by the affine function $z \to c(z - \zeta)$, where $c = \sqrt[\delta-1]{a_\delta}$. With this terminology, we can restate Proposition 2.3 as follows.

Proposition 2.4. *Let p be in normal form. Then Σ_p is infinite if and only if $p(z) = z^\delta$, or equivalently, if $J_p = S^1$. In this case, $\Sigma_p \simeq S^1$.*

We can completely determine the group Σ_p even if it is finite.

Proposition 2.5 ([1, Theorem 5]). *Let p be in normal form. Then the order of Σ_p is equal to the largest integer m such that p can be written in the form $p(z) = z^r Q(z^m)$ for some polynomial Q.*

The tools for the proofs of these facts are the Green and Böttcher functions of p. We generalize Propositions 2.1 and 2.2 in Section 4, and Propositions 2.3 and 2.4 in Section 5. We use Proposition 2.5 to prove a lemma in Section 5.

3. Dynamics of polynomial skew products

In this section, we recall the dynamics of polynomial skew products on $\mathbb{C}^2$ and give the definition of Julia sets.

3.1. **Polynomial skew products.**

A polynomial skew product on $\mathbb{C}^2$ is a polynomial map of the form $f(z,w) = (p(z), q(z,w))$. Let

$$\begin{cases} p(z) = a_\delta z^\delta + a_{\delta-1} z^{\delta-1} + \cdots + a_0, \\ q(z,w) = q_z(w) = b_d(z) w^d + b_{d-1}(z) w^{d-1} + \cdots + b_0(z), \end{cases}$$

and let b_d be a polynomial in z of degree $l \geq 0$. We assume that $\delta \geq 2$ and $d \geq 2$. As in [3], we say that f is nondegenerate if b_d is a nonzero constant.

Let us briefly recall the dynamics of polynomial skew products. Roughly speaking, the dynamics of f consists of the dynamics on the base space and on the fibers. The first component p defines the dynamics on the base space $\mathbb{C}$. Note that f preserves the set of vertical lines in $\mathbb{C}^2$. In this sense, we often use the notation $q_z(w)$ instead of $q(z,w)$. The restriction of f^n to vertical line $\{z\} \times \mathbb{C}$ is viewed as the composition of n polynomials on $\mathbb{C}$, $q_{p^{n-1}(z)} \cdots q_{p(z)} q_z$. Therefore, the n-th iterate of f is written as follows:

$$f^n(z,w) = (p^n(z), Q_z^n(w)),$$

where $Q_z^n(w) = q_{p^{n-1}(z)} \cdots q_{p(z)} q_z(w)$.

3.2. **Green and Böttcher functions.**

It is well known that for a polynomial p, the Green function of p is well defined and useful for studying the dynamics of p. In a similar fashion, we define the fiberwise Green function of f as follows:

$$G_z(w) = \lim_{n \to \infty} d^{-n} \log^+ |Q_z^n(w)|.$$

Favre and Guedj [2] showed that the limit G_z defines a local bounded function on $K_p \times \mathbb{C}$ such that $G_{p(z)}(q_z(w)) = dG_z(w)$. In fact, they used the limit $\lim_{n \to \infty} d^{-n} \log \|Q_z^n(w)\|$, where $\|w\| = |w| + 1$, which coincides with G_z on $K_p \times \mathbb{C}$. However, it is not continuous in general. If $b_d^{-1}(0) \cap K_p = \emptyset$, then it is continuous on $K_p \times \mathbb{C}$. We define $K_z =$

$\{w : G_z(w) = 0\}$, which is nonempty for any z in K_p. To describe G_z more precisely, define

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} \log |b_d(p^n(z))|.$$

It belongs to $L^1(\mu_p)$, where μ_p is the Green measure of p. For fixed z in $K_p \setminus \{\Phi = -\infty\}$, the function G_z is nonnegative, continuous and subharmonic on $\mathbb{C}$. More precisely, it is harmonic on $\mathbb{C} \setminus K_z$ and $G_z(w) = \log |w| + \Phi(z) + o_z(1)$ as $w \to \infty$. This is the Green function for the compact set K_z with a pole at infinity. We remark that $K_p \setminus \{\Phi = -\infty\}$ is forward invariant under p; that is, $p(K_p \setminus \{\Phi = -\infty\}) \subset K_p \setminus \{\Phi = -\infty\}$.

The fiberwise Green function G_z induces the fiberwise Böttcher function φ_z, which is useful to investigate the symmetries of Julia sets.

Lemma 3.1. *For every z in $K_p \setminus \{\Phi = -\infty\}$, there exists a unique conformal function φ_z defined near infinity such that*

(i) $\varphi_z(w) = w + O_z(1)$ *as* $w \to \infty$,
(ii) $\log |c_z \varphi_z(w)| = G_z(w)$, *where* $c_z = \exp(\Phi(z))$,
(iii) $\varphi_{p(z)}(q_z(w)) = b_d(z)(\varphi_z(w))^d$.

3.3. **Julia sets.** In this paper, we consider the following Julia set:

$$J_f = \bigcup_{z \in J_p} \{z\} \times \partial K_z.$$

Here we define $\partial K_z = \emptyset$ if $K_z = \mathbb{C}$. We call ∂K_z the fiberwise Julia set. Hence J_f is the union of the fiberwise Julia sets over the base Julia set J_p. It follows that J_f is forward invariant under f; that is, $f(J_f) \subset J_f$. If $b_d^{-1}(0) \cap J_p = \emptyset$, then J_f is completely invariant under f. Moreover, J_f is compact if and only if $b_d^{-1}(0) \cap J_p = \emptyset$.

The following subset of J_p plays an important role in the proofs:

$$J_p^* = J_p \setminus \{\Phi = -\infty\}.$$

Note that J_p^* is dense in J_p because it contains most periodic points. For any z in J_p^*, the limits G_z and φ_z are well defined. In addition, J_p^* is forward invariant under p, and more precisely, $J_p^* \setminus p(J_p^*) \subset p(b_d^{-1}(0))$.

There is another Julia set of f that might be appropriately called the Julia set of f. Favre and Guedj [2] showed that the closure

$$\overline{\bigcup_{z \in J_p^*} \{z\} \times \partial K_z}$$

coincides with the support of the Green measure of f. Similar to J_f, this Julia set is compact if and only if $b_d^{-1}(0) \cap J_p = \emptyset$.

Remark 3.2. *The same results hold for the symmetries of the last Julia set if* $b_d^{-1}(0) \cap J_p = \emptyset$*, or if it holds that* K_z *contains the restriction of the last Julia set to* $\{z\} \times \mathbb{C}$ *for any periodic point* z *in* J_p^**.*

4. Symmetries of Julia sets

In this section, we consider suitable symmetries of the Julia set of a polynomial skew product f. As a symmetry, we consider a polynomial automorphism of the form $\gamma(z,w) = (\gamma_1(z), \gamma_2(z,w))$ that preserves J_f. Since γ_1 is conformal, $\gamma_1(z) = c_1 z + c_2$, where c_1 and c_2 are complex numbers. Since J_p is compact, $|c_1| = 1$. Since γ_2 is conformal on each fiber, $\gamma_2(z,w) = c_3 w + c_4(z)$, where c_3 is a complex number and c_4 is a polynomial in z. Since K_z is compact for some z in J_p, it follows that $|c_3| = 1$. Therefore, we define the group of the symmetries of J_f as

$$\Gamma_f = \{\gamma \in S : \gamma(J_f) = J_f\},$$

where

$$S = \left\{ \gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} c_1 z + c_2 \\ c_3 w + c_4(z) \end{pmatrix} : |c_1| = |c_3| = 1 \right\}.$$

Let us denote γ in Γ_f by $(\sigma(z), \gamma_z(w))$. Since σ preserves J_p, it follows that σ belongs to Σ_p. By definition, $\gamma_z(\partial K_z) = \partial K_{\sigma(z)}$ and so $\gamma_z(K_z) = K_{\sigma(z)}$ for any z in J_p.

4.1. Centroids.

As defined in Section 2, we define the centroids of f as

$$\zeta = \frac{-a_{\delta-1}}{\delta a_\delta} \quad \text{and} \quad \zeta_z = \frac{-b_{d-1}(z)}{d b_d(z)}.$$

Although ζ is a constant, ζ_z is a rational function in z, If f is nondegenerate, then ζ_z is a polynomial.

The fiberwise Böttcher function φ_z relates to the centroid ζ_z. The following proposition follows from (i) and (iii) in Lemma 3.1.

Lemma 4.1. *It follows that* $\varphi_z(w) = w - \zeta_z + o_z(1)$ *for any* z *in* J_p^**.*

We first show that a symmetry γ is birationally conjugate to a rotational product, which generalizes Proposition 2.1.

Proposition 4.2. *For any* γ *in* Γ_f*, there exist* μ *and* ν *in* S^1 *such that*

$$\gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta) + \zeta \\ \nu(w - \zeta_z) + \zeta_{\sigma(z)} \end{pmatrix},$$

where $\sigma(z) = \mu(z - \zeta) + \zeta$ *belongs to* Σ_p*.*

Proof. Let us denote γ in Γ_f by $(\sigma(z), \gamma_z(w))$. It follows from Proposition 2.1 that $\sigma(z) = \mu(z-\zeta)+\zeta$ for some μ in S^1.

Fix any z in $J_p^* \setminus \{b_d(\sigma(z)) = 0\}$. It then follows from Lemma 4.1 that

$$\varphi_z(w) = w - \zeta_z + o_z(1), \tag{1}$$

since z belongs to J_p^*. Although $K_{\sigma(z)}$ is compact, we do not know whether $G_{\sigma(z)}$ is the Green function of $K_{\sigma(z)}$. However, $\hat{G}_{\sigma(z)} := G_z \gamma_z^{-1}$ is so. Let $\hat{\varphi}_{\sigma(z)}$ be the fiberwise Böttcher function corresponding to $\hat{G}_{\sigma(z)}$. We show that

$$\hat{\varphi}_{\sigma(z)}(w) = w - \zeta_{\sigma(z)} + o_z(1). \tag{2}$$

Define $\hat{G}_{p\sigma(z)} = \hat{G}_{\sigma^\delta p(z)} := G_{p(z)}\gamma_{p(z)}^{-1}\gamma_{\sigma p(z)}^{-1}\cdots\gamma_{\sigma^{\delta-1}p(z)}^{-1}$, which is also the Green function of the compact set $K_{\sigma p(z)}$. Let $\hat{\varphi}_{p\sigma(z)}$ be the fiberwise Böttcher function corresponding to $\hat{G}_{p\sigma(z)}$. It follows from the assumption $b_d(\sigma(z)) \neq 0$ that $q_{\sigma(z)}$ maps $K_{\sigma(z)}$ onto $K_{p\sigma(z)}$ and that $\hat{\varphi}_{p\sigma(z)}(q_{\sigma(z)}(w)) = b_d(\sigma(z))(\hat{\varphi}_{\sigma(z)}(w))^d$. Hence we get equation (2).

Now, let us induce the required formula. Since $\hat{G}_{\sigma(z)}\gamma_z$ and G_z are the Green functions for K_z with a pole at infinity, it follows from the uniqueness property of the Green functions that $\hat{G}_{\sigma(z)}\gamma_z = G_z$. Hence Lemma 3.1 implies that $c_{\sigma(z)}\hat{\varphi}_{\sigma(z)}(\gamma_z(w)) = \nu_z c_z \varphi_z(w)$ for some ν_z in S^1. Because $\gamma_z(w) = c_3 w + c_4(z)$, the ratio of $\nu_z c_z$ and $c_{\sigma(z)}$ is equal to the constant c_3. Hence, replacing c_3 by ν, we have the equation

$$\hat{\varphi}_{\sigma(z)}(\gamma_z(w)) = \nu\varphi_z(w).$$

Combining this equation and equations (1) and (2) yields

$$\gamma_z - \zeta_{\sigma(z)} + o_z(1) = \nu(w - \zeta_z + o_z(1)).$$

Therefore, by comparing the regular terms in this equation, it follows that $\gamma_z(w) = \nu(w-\zeta_z) + \zeta_{\sigma(z)}$ on $J_p^* \setminus \{b_d(\sigma(z)) = 0\} \times \mathbb{C}$.

Finally, we extend this equation to $\mathbb{C}^2$. The left side $\gamma_z(w)$ is holomorphic on $\mathbb{C}^2$ and the right side $\nu(w-\zeta_z)+\zeta_{\sigma(z)}$ is holomorphic on $\{z : b_d(z)b_d(\sigma(z)) \neq 0\} \times \mathbb{C}$. By identity theorem of holomorphic functions on horizontal lines, the equation above holds on the complement of the finite fibers. By Riemann's removable singularity theorem of holomorphic functions on horizontal lines, $\nu(w-\zeta_z)+\zeta_{\sigma(z)}$ is also holomorphic on $\mathbb{C}^2$. Indeed, it is a polynomial on $\mathbb{C}^2$. Therefore, $\gamma_z(w) = \nu(w-\zeta_z)+\zeta_{\sigma(z)}$ on $\mathbb{C}^2$. □

Corollary 4.3. *It follows that σ preserves the set $\{z \in J_p : \zeta_z = \infty\}$, where σ is the first component of γ in Γ_f.*

By Proposition 4.2, we can identify Γ_f with a subgroup of the torus:

$$\Gamma_f = \left\{ \gamma_{\mu,\nu} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix} : \gamma_{\mu,\nu}(J_f) = J_f \right\}$$

$$\simeq \{(\mu,\nu) \in S^1 \times S^1 : \gamma_{\mu,\nu} \in \Gamma_f\} \subset S^1 \times S^1.$$

We use the notation $=$ instead of $\simeq$ hereafter. By definition, Γ_f is a subgroup of $\Sigma_p \times S^1$. More practically, the birational map $(z,w) \to (z-\zeta, w-\zeta_z)$ conjugates the symmetry $\gamma_{\mu,\nu}$ in Γ_f to a rotational product $\tilde{\gamma}(z,w) = (\mu z, \nu w)$.

4.2. **Normal form.** As in Section 2, we say that f is in normal form if p and b_d are monic and $a_{\delta-1}$ and b_{d-1} are the constant 0. Roughly speaking, the normality of f is equivalent to those of p and q_z. Hence if f is in normal form, then the centroids are at the origin.

Unlike the cases of polynomials and nondegenerate polynomial skew products, we may not assume that f is in normal form without loss of generality. However, we can normalize f to a rational map as follows. Define $h(z,w) = (c_1(z-\zeta), c_2(w-\zeta_z))$, where $c_1^{\delta-1}$ is equal to a_δ, the coefficient of the leading term of p, and $c_1^l c_2^{d-1}$ is equal to the coefficient of the leading term of b_d. Then h is a birational map. Let $\tilde{f}$ be the conjugation of f by h: $hf = \tilde{f}h$. This rational map satisfies all conditions in the definition of normality. Hence we call $\tilde{f}$ the normalized rational skew product of f, which appears in Section 5.2.

4.3. **Functional equations.** Under some assumptions, we characterize Γ_f by functional equations including the iterates of f, which generalizes Proposition 2.2.

Although the group Σ_p of a polynomial p is characterized by the unique equation $p\sigma = \sigma^\delta p$, our characterization of Γ_f needs infinitely many equations as in [3, Lemma 3.2]. We saw in Proposition 4.2 that γ in Γ_f can be written as

$$\gamma \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu(z-\zeta)+\zeta \\ \nu(w-\zeta_z)+\zeta_{\sigma(z)} \end{pmatrix}.$$

Thus define $\mathcal{E}_f = \{\gamma \in S : f^n\gamma = \gamma_n f^n \text{ for } \forall n \geq 1\}$, where

$$\gamma_n \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \mu^{\delta^n}(z-\zeta)+\zeta \\ \mu^{l_n}\nu^{d^n}(w-\zeta_{p^n(z)})+\zeta_{p^n(\sigma(z))} \end{pmatrix} \text{ and } l_n = \frac{\delta^n - d^n}{\delta - d} l.$$

Unlike the nondegenerate case, we need some assumptions for Γ_f to coincide with $\mathcal{E}_f$, which may be removable.

We first provide a lemma about certain symmetries of the polynomial b_d. In the proofs of Lemma 4.1 and Proposition 4.2, we used the following commutative diagram:

$$\begin{tikzcd}
K^c_{p(z)} \arrow[d, "\varphi_{p(z)}"'] & K^c_z \arrow[l, "q_z"'] \arrow[r, "\gamma_z"] \arrow[d, "\varphi_z"] & K^c_{\sigma(z)} \arrow[r, "q_{\sigma(z)}"] \arrow[d, "\hat{\varphi}_{\sigma(z)}"] & K^c_{p\sigma(z)} \arrow[d, "\hat{\varphi}_{p\sigma(z)}"] \\
\mathbb{C} & \mathbb{C} \arrow[l, "b_d(z)w^d"'] \arrow[r, "\nu w"] & \mathbb{C} \arrow[r, "b_d(\sigma(z))w^d"] & \mathbb{C}
\end{tikzcd}$$

for any z in $J_p^* \setminus \{b_d(\sigma(z)) = 0\}$. Although φ_z may not be defined on $K_z^c = \mathbb{C} \setminus K_z$, it is defined near infinity. Since $p\sigma = \sigma^\delta p$, it follows that $\gamma_{\sigma^{\delta-1}p(z)} \cdots \gamma_{\sigma p(z)} \gamma_{p(z)}(K_{p(z)}) = K_{\sigma^\delta p(z)} = K_{p\sigma(z)}$. Therefore, $|b_d(\sigma(z))(\nu w)^d| = |b_d(z) w^d|$ and so we have the following lemma.

Lemma 4.4. *It follows that $|b_d(\sigma(z))| = |b_d(z)|$ for any symmetry σ and for any z in $J_p^* \setminus \{b_d(\sigma(z)) = 0\}$, where σ is the first component of γ in Γ_f.*

We use this lemma to prove the main theorems in the next section. It is natural to ask whether the equation $b_d(\sigma(z)) = \mu^l b_d(z)$ holds or not, where l is the degree of b_d. In the following proposition, we assume some conditions that guarantee this equation.

Proposition 4.5. *If p is in normal form and $b_d(z) = z^l$, then $\Gamma_f \subset \mathcal{E}_f$. Moreover, σ preserves J_p^*, where σ is the first component of γ in Γ_f.*

Proof. Let us denote γ in Γ_f by $(\sigma(z), \gamma_z(w))$. Although $\sigma(z) = \mu z$ and $b_d(z) = z^l$, we use original notations σ and b_d. It follows from Proposition 2.2 that $p\sigma = \sigma^\delta p$ and so $p^n \sigma = \sigma^{\delta^n} p^n$; that is, $p^n(\sigma(z)) = \mu^{\delta^n}(p^n(z) - \zeta) + \zeta$.

We first show that $f\gamma = \gamma_1 f$. Fix any z in $J_p^* \setminus \{b_d(\sigma(z)) b_d(p\sigma(z)) = 0\}$. It follows from the commutative diagram above that $\varphi_{p(z)}(q_z) = b_d(z)\varphi_z^d$ and $\hat{\varphi}_{p\sigma(z)}(q_{\sigma(z)}\gamma_z) = b_d(\sigma(z))\nu^d \varphi_z^d$. Thus $b_d(z)\hat{\varphi}_{p\sigma(z)}(q_{\sigma(z)}\gamma_z) = b_d(\sigma(z))\nu^d \varphi_{p(z)}(q_z)$, which implies that

$$b_d(z)(q_{\sigma(z)}(\gamma_z(w)) - \zeta_{p\sigma(z)}) = b_d(\sigma(z))\nu^d (q_z(w) - \zeta_{p(z)})$$

on $J_p^* \setminus \{b_d(\sigma(z)) b_d(p\sigma(z)) = 0\} \times \mathbb{C}$. As in the proof of Proposition 4.2, we can extend this equation to $\mathbb{C}^2$. By assumption,

$$b_d(\sigma(z)) = \mu^l b_d(z).$$

Therefore, $q_{\sigma(z)}(\gamma_z(w)) = \mu^l \nu^d (q_z(w) - \zeta_{p(z)}) + \zeta_{p\sigma(z)}$.

Next, we show that $f^n \gamma = \gamma_n f^n$ for any $n \geq 2$. Let $D = \{z : b_d(z) b_d(\sigma(z)) b_d(p\sigma(z)) = 0\}$. Although we do not know whether γ_1 belongs to Γ_f, similar arguments as above induce the equation $Q^2_{\sigma(z)}(\gamma_z(w))$

$= \mu^{(\delta+d)l}\nu^{d^2}(Q_z^2(w) - \zeta_{p^2(z)}) + \zeta_{p^2\sigma(z)}$ for any z in $J_p^* \setminus \cup_{j=0}^{1} p^j(D)$. Similarly, we have the equation $Q_{\sigma(z)}^n(\gamma_z(w)) = \mu^{l_n}\nu^{d^n}(Q_z^n(w) - \zeta_{p^n(z)}) + \zeta_{p^n\sigma(z)}$ for any z in $J_p^* \setminus \cup_{j=0}^{n-1} p^j(D)$, which extends to $\mathbb{C}^2$.

Finally, we show that σ preserves J_p^*. It follows from the equation $b_d(\sigma(z)) = \mu^l b_d(z)$ that $b_d(p^n\sigma(z)) = \mu^{l\delta^n} b_d(p^n(z))$. Thus $\Phi(\sigma(z)) = \Phi(z)$ and so σ preserves $J_p^* = J_p \setminus \{\Phi = 0\}$. □

With a slight change in the proof, we can replace the assumption in this proposition with the assumption that f is in normal form and q is not divisible by any polynomial in z.

The following corollary of Proposition 4.5 is useful to determine Γ_f for a given map f. In fact, we use this corollary to calculate the groups of symmetries for some examples in Section 4.4 and to prove the main theorems in Sections 5.1 and 5.2.

Corollary 4.6. *If f is in normal form and $b_d(z) = z^l$, then*

$$q(\mu z, \nu w) = \mu^l \nu^d q(z, w)$$

for any $\gamma(z, w) = (\mu z, \nu w)$ in Γ_f.

For the inverse inclusion, we have the following statement.

Proposition 4.7. *If $b_d^{-1}(0) \cap J_p = \emptyset$ or $b_{d-1}(z) \equiv 0$, then $\Gamma_f \supset \mathcal{E}_f$.*

Proof. Let us denote γ in $\mathcal{E}_f$ by $(\sigma(z), \gamma_z(w))$. It then follows that $G_p(\sigma(z)) = G_p(z)$ and so $\sigma(J_p) = J_p$.

First, we consider the case $b_{d-1}(z) \equiv 0$. Then the equation $f^n\gamma = \gamma_n f^n$ implies the equation $Q_{\sigma(z)}^n \gamma_z = \mu^{l_n}\nu^{d^n} Q_z^n$ for any z in $\mathbb{C}$. Hence $G_{\sigma(z)}\gamma_z = G_z$ and so $\gamma_z(K_z) = K_{\sigma(z)}$ for any z in K_p. Therefore, $\gamma(J_f) = J_f$.

Next, we consider the case $b_d^{-1}(0) \cap J_p = \emptyset$. The equation $f^n\gamma = \gamma_n f^n$ implies the equation $Q_{\sigma(z)}^n(\gamma_z(w)) = \mu^{l_n}\nu^{d^n}(Q_z^n(w) - \zeta_{p^n(z)}) + \zeta_{p^n\sigma(z)}$. Since $b_d^{-1}(0) \cap J_p = \emptyset$, the centroids $\zeta_{p^n(z)}$ and $\zeta_{p^n\sigma(z)}$ are uniformly bounded for any z in J_p and for any $n \geq 1$. Hence $G_{\sigma(z)}\gamma_z = G_z$ for any z in J_p, which completes the proof. □

Combining Propositions 4.5 and 4.7, we get sufficient conditions for Γ_f to coincide with $\mathcal{E}_f$.

Corollary 4.8. *Assume that f satisfies one of the following conditions: (i) f is in normal form and q is not divisible by any polynomial in z, (ii) f is in normal form and $b_d(z) = z^l$, (iii) $p(z) = z^\delta$ and $b_d(z) = z^l$. Then $\Gamma_f = \mathcal{E}_f$. In particular, Γ_f is compact, and γ_n belongs to Γ_f for any $n \geq 1$ if γ belongs to Γ_f.*

4.4. **Examples.** Let us provide some examples of the groups of symmetries. For a map f of examples below, we look for Γ_f as follows. The symmetries of J_f, i.e., the pairs of the two numbers in the torus, have to satisfy the equation in Corollary 4.6. Moreover, by Corollary 4.8, the pair of two numbers satisfying the equation in Corollary 4.6 belongs to Γ_f if and only if it satisfies the infinitely many equations in $\mathcal{E}_f$.

Example 4.9. *Let $f(z,w) = (z^3, zw^2 + z)$. Then $\Gamma_f = \{(\mu,\nu) : \mu^2 = \nu^2 = 1\} = \{(1,1), (-1,\ -1), (1,-1), (-1,1)\}$. Moreover, let $g(z,w) = (z^3, zw^2 + 2z^2w + z)$. Then it is conjugate to f by $h(z,w) = (z, w - z)$: $hf = gh$. Hence $\Gamma_g = \{(z,w), (-z,-w), (z,-w-2z), (-z, w+2z)\}$.*

Example 4.10. *Let $f(z,w) = (z^2 - 1, z^2w^2)$. Then $\Gamma_f = \{\pm 1\} \times S^1$.*

Example 4.11. *Let $f(z,w) = (z^3, zw^2 + z^3)$. Then $\Gamma_f = \{(\mu,\nu) : \mu^2 = \nu^2 \in S^1\}$. Moreover, f is semiconjugate to $f_0(z,w) = (z^3, w^2+1)$ by $\pi(z,w) = (z, zw) : \pi f_0 = f\pi$.*

Example 4.12. *Let $f(z,w) = (z^2, z^3w^5 + zw^3 + w^2)$. Then $\Gamma_f = \{(\mu,\nu) : \mu = \nu^{-1} \in S^1\}$. Moreover, f is semiconjugate to $f_0(z,w) = (z^2, w^5 + w^3 + w^2)$ by $\pi(z,w) = (z, z^{-1}w) : \pi f_0 = f\pi$.*

Although the map f below does not satisfy any condition in Corollary 4.8, it again follows that $\Gamma_f = \mathcal{E}_f$.

Example 4.13. *Let $f(z,w) = (z^2, (z^l - 1)w^2)$, where $l \geq 1$. It follows from Proposition 4.7 that $\Gamma_f \supset \{\mu^l = 1\} \times S^1$. Let $\sigma(z) = \mu z$ be the first component of γ in Γ_f. Lemma 4.4 implies that $|\mu^l z^l - 1| = |z^l - 1|$ on a dense subset of $J_p = S^1$. Hence $\mu^l = 1$ and so $\Gamma_f = \{\mu^l = 1\} \times S^1$.*

In particular, the groups of symmetries of Examples 4.10, 4.11, 4.12 and 4.13 are infinite.

5. Infinite symmetries

In this section, we classify the polynomial skew products whose Julia sets have infinitely many symmetries. We first show that these maps in normal form are classified into four types in Section 5.1. We then remove the assumption of normality and show that the normalized rational skew products of these maps are also classified into four types in Section 5.2.

These maps include polynomial skew products that are semiconjugate to polynomial products such as those given in Examples 4.11 and 4.12. The following lemma gives a sufficient condition of a polynomial map $(z^\delta, q(z,w))$ to be semiconjugate to a polynomial product.

Lemma 5.1. *Let $q(z,w)$ be a polynomial. If there exist nonzero integers r and s and positive integer δ such that $q(z^r, z^s w) = z^{s\delta} q(1,w)$, then $(z^\delta, q(z,w))$ is semiconjugate to $(z^\delta, q(1,w))$ by $\pi(z,w) = (z^r, z^s w)$,*

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "{(z^\delta, q(1,w))}"] \arrow[d, "\pi"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\
\mathbb{C}^2 \arrow[r, "{(z^\delta, q(z,w))}"] & \mathbb{C}^2.
\end{tikzcd}$$

Moreover, q consists of the terms of the form $c_j z^{n_j} w^j$, where c_j is a constant and $n_j = (\delta - j)s/r$ for $0 \le j \le \deg q$.

The former statement of this lemma holds even if q is a rational function; we apply this lemma for the normalized rational skew products in Section 5.2. The equation $n_j = (\delta - j)s/r$ follows from the identity $rn_j + sj = s\delta$.

5.1. **Classification of the maps in normal form.** We first assume that polynomial skew products are in normal form and classify the maps whose Julia sets have infinitely many symmetries.

Theorem 5.2. *Let f be in normal form. Then Γ_f is infinite if and only if one of the following holds*:

(i) $f(z,w) = (z^\delta, z^l w^d)$,
(ii) $f(z,w) = (z^\delta, q(w))$,
(iii) $f(z,w) = (p(z), b_d(z) w^d)$,
(iv) $f(z,w) = (z^\delta, q(z,w))$ *and it is semiconjugate to* $(z^\delta, q(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$ *for some nonzero coprime integers r and s. If $l = 0$, then $\delta = d$ and $s/r > 0$. If $l \neq 0$, then $\delta \neq d$ and $s/r = l/(\delta - d)$.*

To avoid overlap, we assume that $q(w) \neq w^d$ in (ii), $p(z) \neq z^\delta$ or $b_d(z) \neq z^l$ in (iii), and $q(z,w) \neq b_d(z) w^d$ in (iv).

Proof. If f is one of the four maps above, then clearly Γ_f is infinite; see Proposition 5.4 below for details.

Let us show that if Γ_f is infinite, then f is one of the forms from (i) to (iv). Let $\Gamma = \Gamma_f$ be infinite. Then we have three cases: $\pi_z(\Gamma)$ is finite, $\pi_w(\Gamma)$ is finite, or $\pi_z(\Gamma)$ and $\pi_w(\Gamma)$ are both infinite. Here π_z and π_w are the projections to the z and w coordinates, respectively.

We first show that $\pi_z(\Gamma)$ being finite induces (iii). If $\pi_z(\Gamma)$ is finite, then Γ contains $\{1\} \times S^1$ since it is an infinite subgroup of $S^1 \times S^1$ and since J_z is closed for any z in J_p. Thus J_z is a circle C_z about the origin for any z in J_p^*. Fix any z in J_p^*. For some number R_z depending

on z, a map $R_z w^d$ maps C_z to $C_{p(z)}$. Hence $G_{p(z)}(q_z(w)) = dG_z(w) = G_{p(z)}(R_z w^d)$ and so

$$\frac{\varphi_{p(z)}(q_z(w))}{b_d(z)} = \varphi_z(w)^d = \frac{\varphi_{p(z)}(R_z w^d)}{R_z}.$$

Therefore, $q_z(w) = b_d(z)w^d$ on $J_p^* \times \mathbb{C}$. It follows that $q_z(w) = b_d(z)w^d$ on $\mathbb{C}^2$ from the uniqueness theorem of holomorphic functions on horizontal lines, which satisfies (iii).

Let $\pi_z(\Gamma)$ be infinite. Then $p(z) = z^\delta$ and $J_p = S^1$. Moreover, we show that $b_d(z) = z^l$. By Lemma 4.4, for any z in $J_p^* \setminus \{b_d(\sigma(z)) = 0\}$, it follows that $|b_d(\mu z)| = |b_d(z)|$ for any μ in a dense subset $\pi_z(\Gamma)$ of S^1. Thus b_d maps a circle about the origin to a circle about the origin because it is polynomial. Therefore, $b_d(z) = z^l$ since it is monic. In particular, $b_d^{-1}(0) \cap J_p = \emptyset$. Thus J_f and Γ_f are compact.

We next show that $\pi_w(\Gamma)$ being finite induces (ii). If $\pi_w(\Gamma)$ is finite, then Γ contains $S^1 \times \{1\}$ since it is compact. Thus fiverwise Julia sets over J_p are all the same Julia set J, and so the polynomials of the form $q_z(w)$ on fibers over J_p differ only in terms of the symmetries of J. Because the number of the symmetries of J is finite and because J_p is connected, the polynomials on fibers over J_p are all the same. From the uniqueness theorem of holomorphic functions on horizontal lines, it follows that the polynomials on fibers over $\mathbb{C}$ are all the same, which satisfies (ii).

Now we assume that $\pi_z(\Gamma)$ and $\pi_w(\Gamma)$ are both infinite. Since $b_d(z) = z^l$, if q_z consists of the unique term $b_d(z)w^d$, then $f(z,w) = (z^\delta, z^l w^d)$, which satisfies (i). We show that the assumption $q_z(w) \neq b_d(z)w^d$ implies (iv). It follows from Corollary 4.6 that $q(\mu z, \nu w) = \mu^l \nu^d q(z,w)$ for any (μ,ν) in Γ. Therefore, if q contains the term $z^{n_j} w^{m_j}$ with a nonzero coefficient for $m_j < d$, then μ and ν are related by $\mu^{n_j}\nu^{m_j} = \mu^l \nu^d$. The equations $\mu^{n_j - l} = \nu^{d - m_j}$ and $\mu^{n_i - l} = \nu^{d - m_i}$ implies that $\mu^{(n_j - l)(d - m_i)} = \mu^{(n_i - l)(d - m_j)}$, which implies that $(n_j - l)(d - m_i) = (n_i - l)(d - m_j)$ since we may assume that $\mu^n \neq 1$ for any integer $n \neq 0$. Thus the ratio of $n_j - l$ and $d - m_j$ does not depend on j. Let s and r be nonzero coprime integers whose ratio is equal to that of $n_j - l$ and $d - m_j$,

$$\frac{n_j - l}{d - m_j} = \frac{n_i - l}{d - m_i} =: \frac{s}{r},$$

which induces the equation $q(z^r, z^s w) = z^{rl+sd} q(1,w)$. Note that if $l = 0$, then $s/r > 0$ since $m_j < d$.

Moreover, we show that if $l = 0$, then $\delta = d$. It follows from Corollary 4.8 that (μ^δ, ν^d) belongs to Γ for any (μ,ν) in Γ. Thus $(1, \nu^{d-\delta}) =$

$(\mu^\delta, \nu^d) - (\mu^\delta, \nu^\delta)$ belongs to Γ. Since $(1, \nu^{(d-\delta)n})$ belongs to Γ for any integer n, if we suppose that $\delta \neq d$, then it follows that Γ contains $\{1\} \times S^1$, which implies that $q_z(w) = w^d$. This contradicts the assumption $q_z(w) \neq b_d(z)w^d$. Therefore, $\delta = d$ and so $q(z^r, z^s w) = z^{s\delta} q(1, w)$, which implies (iv) as in Lemma 5.1. Let us show that if $l \neq 0$, then $\delta \neq d$ and $s/r = l/(\delta - d)$. Let a_j be an integer such that $n_j - l = a_j s$ and $d - m_j = a_j r$ and let a be the greatest common divisor of a_j. By definition, we obtain the inclusion relation $\Gamma \subset \{\mu^{as} = \nu^{ar}\}$. It follows from Corollary 4.8 that $(\mu^\delta, \mu^l \nu^d)$ belongs to Γ for any (μ, ν) in Γ. Thus $(1, \mu^l \nu^{d-\delta}) = (\mu^\delta, \mu^l \nu^d) - (\mu^\delta, \nu^\delta)$ belongs to Γ. Since $\Gamma \subset \{\mu^{as} = \nu^{ar}\}$, we have $(\mu^l \nu^{d-\delta})^{ar} = 1^{as} = 1$. Substituting $\nu = \mu^{as/ar}$, we get $\mu^{a\{rl + s(d-\delta)\}} = 1$ and so $rl + s(d - \delta) = 0$. Since $l \neq 0$, we conclude that $\delta \neq d$ and $s/r = l/(\delta - d)$. Therefore, $q(z^r, z^s w) = z^{s\delta} q(1, w)$, which implies (iv) as in Lemma 5.1. □

In [2, Section 6.2], Favre and Guedj studied the dynamics of polynomial skew products of the form (iii). To add more informations about polynomial skew products whose Julia sets have infinitely many symmetries, we provide the following lemma and proposition.

Lemma 5.3. *Let f be of the form (iv) in Theorem 5.2 and let a be the greatest common divisor in the proof of Theorem 5.2. Then the order of $\Sigma_{q(1,w)}$ is equal to the absolute value of ar.*

Proof. Let $q(1, w) = w^d + c_1 w^{m_1} + c_2 w^{m_2} + \cdots + c_{t-1} w^{m_{t-1}} + c_t w^{m_t}$, where $d > m_1 > m_2 > \cdots > m_t$. Since $q(1, w) = w^{m_t}(w^{d-m_t} + c_1 w^{m_1 - m_t} + c_2 w^{m_2 - m_t} + \cdots + c_{t-1} w^{m_{t-1} - m_t} + c_t)$, it follows from Proposition 2.5 that the greatest common divisor of $\{d - m_t, m_1 - m_t, \cdots, m_{t-1} - m_t\}$ is equal to the order of $\Sigma_{q(1,w)}$. Since $d - m_j = a_j r$, where $a_1 < a_2 < \cdots < a_t$, we have $d - m_t = a_t r$ and $m_j - m_t = (a_t - a_j)r$. Denoting by gcd the greatest common divisor, we get gcd $\{m_1 - m_t, m_2 - m_t, \cdots, d - m_t\} =$ gcd $\{(a_t - a_1)r, (a_t - a_2)r, \cdots, a_t r\} = r$ gcd $\{a_t - a_1, a_t - a_2, \cdots, a_t\} = r$ gcd $\{a_1, a_2, \cdots, a_t\} = ra$. □

In Theorem 5.2, we gave a sufficient and necessary condition, in terms of the form of f, for the group Γ_f to be infinite. These conditions can be replaced with the shape of J_f or Γ_f as follows.

Proposition 5.4. *Let f be in normal form. Then Γ_f is infinite if and only if one of the following holds*:

(i) $f(z, w) = (z^\delta, z^l w^d)$,
(ii) $f(z, w) = (z^\delta, q(w))$,
(iii) $f(z, w) = (p(z), b_d(z) w^d)$,

(iv) $f(z,w) = (z^\delta, q(z,w))$ *and it is semiconjugate to* $(z^\delta, q(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$ *for some nonzero coprime integers* r *and* s.

To avoid overlap, we assume that $q(w) \neq w^d$ *in (ii), that* $p(z) \neq z^\delta$ *or* $b_d(z) \neq z^l$ *in (iii), and that* $q(z,w) \neq b_d(z)w^d$ *in (iv). Each condition is equivalent to the following:*

(i)' $J_f = S^1 \times S^1$,
(ii)' $J_f = S^1 \times J$,
(iii)' $J_f = \bigcup_{z \in J_p} \{z\} \times C_z$,
(iv)' $J_f = \bigcup_{z \in S^1} \{z\} \times z^{\frac{s}{r}} J$,

for a one-dimensional Julia set J *which is not* S^1 *and for a circle* C_z *about the origin. More precisely,* J *in (ii)' is* J_q*, a circle* C_z *in (iii)' is* $\{\log|w| = -\Phi(z)\}$*, and* J *in (iv)' is* $J_{q(1,w)}$*. To avoid overlap, we assume in* (iii)*' that* J_p *is not* S^1 *or* C_z *is not all the same over* J_p*. We remark that* J_f *in (i)', (ii)' and (iv)' is compact. Or equivalently,*

(i)" $\Gamma_f = S^1 \times S^1$,
(ii)" $\Gamma_f = S^1 \times \Sigma$,
(iii)" $\Gamma_f = \Sigma \times S^1$,
(iv)" $\Gamma_f = \{(\mu, \nu) \in S^1 \times S^1 : \mu^{as} = \nu^{ar}\}$,

for a finite group Σ *in* S^1 *and for a nonzero integer* a*. In particular,* Γ_f *is compact. More precisely,* Σ *in (ii)" is* Σ_q*, and* Σ *in (iii)" is a subgroup of* Σ_p *which includes* $\Sigma_p \cap \{\sigma(z) = \mu z : b_d(\mu z) = \mu^l b_d(z)\}$*. Finally, the integer* a *in* (iv) *is the greatest common divisor in the proof of Theorem 5.2.*

Proof. As in Theorem 5.2, if Γ_f is infinite, then f is one of the forms from (i) to (iv).

Let us show the equivalency of (i), (i)' and (i)". It is clear that (i) implies (i)', which implies (i)". Let $\Gamma_f = S^1 \times S^1$. Since $\pi_z(\Gamma_f)$ is infinite, $p(z) = z^\delta$ and $b_d(z) = z^l$. Since $\Gamma_f \supset \{1\} \times S^1$, it follows that $q(z,w) = b_d(z)w^d$ as in the proof of Theorem 5.2, which satisfies (i).

Let us show the equivalency of (ii), (ii)' and (ii)". It is clear that (ii) implies (ii)', which implies (ii)". We saw that (ii)" implies (ii) in the proof of Theorem 5.2.

Let us show the equivalency of (iii), (iii)' and (iii)". It is clear that (iii) implies (iii)', which implies (iii)". More precisely, if $p(z) \neq z^\delta$, then $J_p \neq S^1$ and so Σ is finite. If $p(z) = z^\delta$ and $b_d(z) \neq z^l$, then the fiberwise Julia sets are not all the same. Otherwise, $\Gamma_f = S^1 \times S^1$ and so $f(z,w) = (z^\delta, z^l w^d)$, which contradicts the assumption (iii). It again follows that Σ is finite, because Σ being infinite induces that $b_d(z) = z^l$ as the proof of Theorem 5.2. We saw that (iii)" implies (iii) in the proof of Theorem 5.2.

Let us show the equivalency of (iv), (iv)' and (iv)". It is clear that (iv) implies (iv)', which implies (iv)". More precisely, it follows from Lemma 5.3 that the integer a in (iv)" is equal to the greatest common divisor in the proof of Theorem 5.2. Let $\Gamma_f = \{\mu^{as} = \nu^{ar}\}$. Since $\pi_z(\Gamma_f)$ and $\pi_w(\Gamma_f)$ are infinite, f is of form (i) or (iv). Since $\Gamma_f \neq S^1 \times S^1$, we conclude that f is of form (iv). $\square$

5.2. **Classification of normalized rational skew products.** Now we classify the polynomial skew products whose Julia sets have infinitely many symmetries. We saw in Section 4.2 that the birational map $h(z,w) = (c_1(z-\zeta), c_2(w-\zeta_z))$ conjugates f to the normalized rational skew product $\tilde{f}$: $hf = \tilde{f}h$. Note that h also conjugates a symmetry γ, which corresponds to μ and ν, to a rotational product $\tilde{\gamma}(z,w) = (\mu z, \nu w)$. Let $\tilde{f}(z,w) = (\tilde{p}(z), \tilde{q}(z,w))$ and let $\tilde{q}(z,w) = \tilde{b}_d(z)w^d + \tilde{b}_{d-1}(z)w^{d-1} + \cdots + \tilde{b}_0(z)$. Then $\tilde{p}$ and $\tilde{b}_d$ are polynomial and $\tilde{b}_{d-1} \equiv 0$. In fact, $\tilde{p}(z) = c_1\{p(c_1^{-1}z + \zeta) - \zeta\}$ and $\tilde{b}_d(z) = c_2^{-d}\{b_d(c_1^{-1}z + \zeta)\}$. Similarly, let $\tilde{J}_p$ and $\tilde{J}_p^*$ be the conjugations of J_p and J_p^* under the first component of h.

Theorem 5.5. *Let f be a polynomial skew product whose Julia set has infinitely many symmetries. Then the normalized rational skew product $\tilde{f}$ is one of the following:*

(i) $\tilde{f}(z,w) = (z^\delta, z^l w^d)$,
(ii) $\tilde{f}(z,w) = (z^\delta, \tilde{q}(w))$,
(iii) $\tilde{f}(z,w) = (\tilde{p}(z), \tilde{b}_d(z)w^d)$,
(iv) $\tilde{f}(z,w) = (z^\delta, \tilde{q}(z,w))$ *and it is semiconjugate to* $(z^\delta, \tilde{q}(1,w))$ *by* $\pi(z,w) = (z^r, z^s w)$ *for some nonzero coprime integers r and s. If $l = 0$, then $\delta = d$ and $s/r > 0$. If $l \neq 0$, then $\delta \neq d$ and $s/r = l/(\delta - d)$.*

In the cases from (i) to (iii), the maps h and $\tilde{f}$ are polynomial. In the case (iv), if $s/r > 0$ then $\tilde{f}$ is polynomial. To avoid overlap, we assume that $\tilde{q}(w) \neq w^d$ in (ii), $\tilde{p}(z) \neq z^\delta$ or $\tilde{b}_d(z) \neq z^l$ in (iii), and $\tilde{q}(z,w) \neq \tilde{b}_d(z)w^d$ in (iv).

Proof. If f is one of the four maps above, then Γ_f is infinite. Let f be a polynomial skew product whose Julia set has infinitely many symmetries. We show that $\tilde{f}$ is one of the forms from (i) to (iv). Since the proof is similar to that of Theorem 5.2, we only give an outline of the proof.

Let $\Gamma = \Gamma_f$ be infinite. Then we have three cases. First, let $\pi_z(\Gamma)$ be finite; then Γ includes $\{1\} \times S^1$ and so $\tilde{J}_z$ is a circle about the origin for any z in $\tilde{J}_p^*$. Hence $\tilde{q} = \tilde{b}_d(z)w^d$, which satisfies (iii).

If $\pi_z(\Gamma)$ is infinite, then $\tilde{p}(z) = z^\delta$ and $\tilde{J}_p = S^1$. Moreover, it follows from Proposition 4.4 that $\tilde{b}_d(z) = z^l$. In particular, J_f and Γ_f are compact and h is homeomorphism on $J_p \times \mathbb{C}$ since $b_d^{-1}(0) \cap J_p = \emptyset$.

Next, let $\pi_w(\Gamma)$ be finite; then Γ includes $S^1 \times \{1\}$. Hence the normalized fiberwise Julia sets over $\tilde{J}_p$ are all the same. Therefore, $\tilde{q}(z,w) = \tilde{q}(w)$, which satisfies (ii).

Finally, we assume that $\pi_z(\Gamma)$ and $\pi_w(\Gamma)$ are both infinite. If $\tilde{q}(z,w) = \tilde{b}_d(z)w^d$, then $\tilde{f}$ is of the form (i). Let us assume that $\tilde{q}(z,w) \neq \tilde{b}_d(z)w^d$. Since $\tilde{p}(z) = z^\delta$ and $\tilde{b}_d(z) = z^l$, it follows that

$$\tilde{q}(z,w) = z^l w^d + \sum c_j z^{n_j} w^{m_j},$$

where c_j is a nonzero coefficient, n_j could be negative, and $0 \leq m_j < d$. With suitable changes of notations, Proposition 4.5 and Corollary 4.6 holds for $\tilde{f}$ because major arguments are made on fibers over the base Julia set. Thus $\tilde{q}(\mu z, \nu w) = \mu^l \nu^d \tilde{q}(z,w)$ for any (μ, ν) in Γ. The same argument as in the proof of Theorem 5.2 induces that $\tilde{q}(z^r, z^s w) = z^{s\delta}\tilde{q}(1,w)$ for some nonzero coprime integers r and s. It then follows from Lemma 5.1 that $\tilde{f}$ is of the form (iv).

From the commutative diagram $hf = \tilde{f}h$, it follows that if $\tilde{f}$ is one of the forms from (i) to (iii), then ζ_z is a polynomial in z. Hence h is polynomial and so is $\tilde{f}$. Let $\tilde{f}$ be the form of (iv) and assume that $s/r > 0$. If $\tilde{q}$ contains the term $z^{n_j} w^j$ with a nonzero coefficient, then it follows from the identity $rn_j + sj = s\delta$ or Lemma 5.1 that $n_j = (\delta - j)s/r$. By assumption, $\delta \geq d \geq j$ and so $n_j \geq 0$. Therefore, $\tilde{f}$ is polynomial. $\square$

Let f be a polynomial skew product whose Julia set has infinitely many symmetries. Then similar statements as Proposition 5.4 hold for $J_{\tilde{f}}$ and Γ_f.

We end the paper with a remark. Let $f(z,w) = (z^2, z^2w^4 + 4zw^3)$, which is semiconjugate to $f_0(z,w) = (z^2, w^4 + 4w^3)$ by $\pi(z,w) = (z, z^{-1}w)$. Then $h(z,w) = (z, w + z^{-1})$ and $\tilde{f}(z,w) = (z^2, z^2w^4 - 6w^2 + 8z^{-1}w - 2z^{-2})$, which are rational. Note that $\tilde{f}$ is semiconjugate to $\tilde{f}_0(z,w) = (z^2, w^4 - 6w^2 + 8w - 2)$ by π. This example of the form (iv) in Theorem 5.5 shows that the normalized rational map in the theorem is not always polynomial. However, toward a description of the classification in terms of polynomial maps, we may ask that whether a polynomial skew product of the form (iv) in Theorem 5.5 is always polynomially conjugate to another polynomial skew product that is semiconjugate to a polynomial product.

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